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Spectra of PT-symmetric operators and perturbation theory

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Abstract

A criterion is formulated for existence and another for the non-existence of complex eigenvalues for a class of non-self-adjoint operators in Hilbert space invariant under a particular discrete symmetry. Applications to the PT-symmetric Schrödinger operators are discussed.

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1. Introduction and statement of the results

The Schrödinger operators invariant under the combined application of a reflection symmetry operator P and of the (antilinear) complex conjugation operation T are called PT-symmetric. A standard class of such operators has the form $H = H_0 + \mathrm{i}W$ where

1. H_0 is a self-adjoint realization of $-\Delta + V$ on some Hilbert space $L^2(\Omega)$; $\Omega \subset \mathbb{R}^n, n \geqslant 1$; V and W are real multiplication operators.
2. V is P-even, $PV = V$, and W is P-odd: $PW = -W$. P is the parity operation

$$(P\psi)(x) = \psi((-1)^{j_1}x_1, \ldots, (-1)^{j_n}x_n), \qquad \psi \in L^2$$

   where $j_i = 0, 1$; $j_i = 1$ for at least one $1 \leqslant i \leqslant n$.

If T is the involution defined by complex conjugation: $(T\psi)(x) = \overline{\psi}(x)$, one immediately checks that $(PT)H = H(PT)$.

PT-symmetric quantum mechanics (see, e.g., [1–8]) requires the reality of the spectrum of PT-symmetric operators, recently proved, for instance, for the one-dimensional odd anharmonic oscillators [12, 13]. Imposing boundary conditions along complex directions, however, examples of PT-symmetric operators with complex eigenvalues have been constructed [14]. It is therefore an important issue in this context to determine whether or not the spectrum of PT-symmetric Schrödinger operators with standard L^2 boundary conditions at infinity is real. We deal with this problem only in perturbation theory, but we will obtain criteria both for existence of complex eigenvalues (theorem 1.1) and for the reality of the spectrum (theorem 1.2), in even greater generality than the PT symmetry.

Let $\mathcal{H}$ be a Hilbert space with scalar product denoted as $(x|y)$, linear in the first factor and antilinear in the second one, and $H_0 : \mathcal{H} \to \mathcal{H}$ be a closed operator with domain $\mathcal{D} \subset \mathcal{H}$. Let H_1 be an operator in $\mathcal{H}$ with $\mathcal{D}(H_1) \supset \mathcal{D}$. This entails that H_1 is bounded relative to H_0, i.e., there exist $b > 0, a > 0$ such that $\|H_1\psi\| \leqslant b\|H_0\psi\| + a\|\psi\|, \forall\psi \in \mathcal{D}$. We can therefore define on $\mathcal{D}$ the operator family $H_\epsilon := H_0 + \epsilon H_1, \forall\epsilon \in \mathbb{C}$.

We assume the following symmetry properties: there exists a unitary involution $J : \mathcal{H} \to \mathcal{H}$ mapping $\mathcal{D}$ to $\mathcal{D}$, such that

$$JH_0 = H_0^* J, \qquad JH_1 = H_1^* J. \tag{1.1}$$

In other words, J intertwines H_0 and H_1 with the corresponding adjoint operators. Note that

1. the properties $J^2 = 1$ (involution) and $J^* = J^{-1}$ (unitarity) entail $J^* = J$, i.e., self-adjointness of J;
2. the properties (1.1) entail, if $\epsilon \in \mathbb{R}$, $JH_\epsilon = H_\epsilon^* J$; therefore the spectrum $\sigma(H_\epsilon)$ of H_ϵ is symmetric with respect to the real axis if $\epsilon \in \mathbb{R}$;
3. an example of J is the parity operator P.

Let H_0 admit a real isolated eigenvalue λ_0 of multiplicity 2 (both algebraic and geometric, i.e., we assume the absence of Jordan blocks). Let e_1, e_2 be linearly independent eigenvectors, and E_{λ_0} the eigenspace spanned by e_1, e_2. Clearly $JE_{\lambda_0} := E_{\lambda_0}^*$ is the eigenspace of H_0^* corresponding to the eigenvalue $\overline{\lambda}_0$, and hence the sesquilinear form $(u^*|v)$, $u^* \in E_{\lambda_0}^*$, $v \in E_{\lambda_0}$ is non-degenerate. Therefore, we can choose e_1, e_2 in E_{λ_0} in such a way that, writing $u = u_1 e_1 + u_2 e_2$, the quadratic form $Q(u, u) = (Ju|u)$ on E_{λ_0} assumes the canonical form

$$Q(u, u) = \tau_1 u_1^2 + \tau_2 u_2^2, \qquad \tau_1 = \pm 1, \quad \tau_2 = \pm 1. \tag{1.2}$$

Under these circumstances we want to prove the following:

Theorem 1.1. *With the above assumptions and notation, consider the operator family H_ϵ for $\epsilon \in \mathbb{R}$. Denote:*

$$H_{11} = (e_1|H_1 e_1), \qquad H_{22} = (e_2|H_1 e_2), \qquad H_{12} = (e_1|H_1 e_2). \tag{1.3}$$

Then $(e_1|H_1e_1) \in \mathbb{R}$, $(e_2|H_1e_2) \in \mathbb{R}$ and there exists $\epsilon^* > 0$ such that, for $0 < |\epsilon| < \epsilon^*$:

(i) If $\tau_1 \cdot \tau_2 = -1$, and

$$H_{12} \neq 0, \qquad 4|H_{12}|^2 > (H_{11} - H_{22})^2 \tag{1.4}$$

then H_ϵ has a pair of non-real, complex conjugate eigenvalues near λ_0.

(ii) If $\tau_1 \cdot \tau_2 = 1$ then H_ϵ has a pair of real eigenvalues near λ_0.

Remarks

1. The above theorem applies to the PT-symmetric operator family $H_\epsilon = H_0 + \mathrm{i}\epsilon W$, where H_0 and $\mathrm{i}W = H_1$ are as above. Here $J = P$, and hence $PH_0 = H_0 P$, $P(\mathrm{i}\epsilon W) = -(\mathrm{i}\epsilon W)P = (\mathrm{i}\epsilon W)^* P$ so that $JH_\epsilon = H_\epsilon^* J$. In that case, moreover, the second condition of (1.4) is satisfied as soon as $H_{12} \neq 0$ because the P-symmetry of H_0 and the P-antisymmetry of W entail $H_{11} = H_{22} = 0$.
2. The physical relevance of theorem 1.1 is best illustrated by an elementary example. Let $\mathcal{H} = L^2(\mathbb{R}^2)$ and $H_0 : \mathcal{H} \to \mathcal{H}$ be the (self-adjoint) two-dimensional harmonic oscillator with frequencies ω_1, ω_2:

$$H_0 u = -\tfrac{1}{2}\Delta u + \tfrac{1}{2}\left(\omega_1^2 x_1^2 + \omega_2^2 x_2^2\right)u.$$

We have $\sigma(H_0) = \{E_{k_1,k_2}\} := \{k_1\omega_1 + k_2\omega_2 + \frac{\omega_1}{2} + \frac{\omega_2}{2}\}, k_i = 0, 1, 2 \ldots, i = 1, 2$. Let again $H_\epsilon = H_0 + \mathrm{i}\epsilon W, \epsilon \in \mathbb{R}$, with

$$W(x) = \frac{x_1^2 x_2}{1 + x_1^2 + x_2^2}.$$

Then W is bounded relative to H_0, and $PW = -W$ if $Pu(x_1, x_2) = u(x_1, -x_2)$ or $Pu(x_1, x_2) = u(-x_1, -x_2)$. Set $\omega_1 = 1, \omega_2 = 2, k_1 = 2, k_2 = 0$; i.e., we consider the eigenvalue $E_{2,0}$. Then for $|\epsilon| > 0$ small enough H_ϵ has a pair of complex conjugate eigenvalues near $E_{2,0}$.

To see this, remark that $E_{2,0} = E_2(\omega_1) + E_0(\omega_2) = E_0(\omega_1) + E_1(\omega_2)$, where $E_i(\omega_i) = (k + 1/2)\omega_i$ are the eigenvalues of the one-dimensional harmonic oscillators with frequencies $\omega_i, i = 1, 2$. $E_{2,0}$ has multiplicity 2. A basis of eigenfunctions is given by

$$\psi_1(x_1, x_2) = e_2(x_1) f_0(x_2); \qquad \psi_1(x_1, x_2) = e_0(x_1) f_1(x_2).$$

Here e_0, e_2 are the eigenfunctions corresponding to $E_0(1)$ and $E_2(1)$, respectively; f_0, f_1 are the eigenfunctions corresponding to $E_0(2)$ and $E_1(2)$, respectively; note that e_0, e_2 and f_0 are even while f_1 is odd. To first-order perturbation theory, the two eigenvalues $\Lambda_j(\epsilon), j = 1, 2$, of H_ϵ near $E_{2,0}$ are given by

$$\Lambda_j(\epsilon) = E_{2,0} + \mathrm{i}\epsilon\lambda_j$$

where $\lambda_j, j = 1, 2$, are the eigenvalues of the 2×2 matrix

$$(W_{l,k}) = \begin{pmatrix} (\psi_1|W\psi_1) & (\psi_1|W\psi_2) \\ (\psi_2|W\psi_1) & (\psi_2|W\psi_2) \end{pmatrix}.$$

Now ψ_1 is even, ψ_2 is odd. Therefore, $\tau_1 \cdot \tau_2 = -1$. Moreover, since W is odd: $(\psi_1|W\psi_1) = (\psi_2|W\psi_2) = 0, (\psi_2|W\psi_1) = (\psi_1, W\psi_2) := w > 0$. Therefore $\lambda_j = \pm w$ and $\Lambda_j(\epsilon) = E_{2,0} \pm \mathrm{i}\epsilon w$. Hence, the conditions of theorem 1.1(i) are satisfied and for ϵ small enough H_ϵ has a pair complex conjugate eigenvalues near $E_{2,0}$.

3. The result of theorem 1.1 remains true under the following more general conditions: under the above assumptions on H_0 and H_1 let H_0 admit two real, simple eigenvalues E_1, E_2. Let $d := E_2 - E_1$ be their relative distance; $D := \mathrm{dist}[(\sigma(H_0) \setminus \{E_2, E_1\}), \{E_2, E_1\}]$ their distance from the rest of the spectrum; ψ_1, ψ_2 the corresponding eigenvectors, all other notation being the same. Then if d/D is small enough the same conclusion of theorem 1.1 holds provided $|\epsilon H_{12}| > \frac{d}{2}$. We will sketch the proof of this statement after the proof of theorem 1.1.
4. *Example*. Odd perturbations of quantum mechanical double wells: existence of complex eigenvalues.

   Let $\mathcal{H} = L^2(\mathbb{R}), H_0(\hbar) = -\hbar^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2(1+x)^2, D(H_0) = H^2(\mathbb{R}) \cap L^2_4(\mathbb{R}), W(x) \in L^\infty_{loc}(\mathbb{R}), |W(x)| \leqslant Ax^4, |x| \to \infty, W(-x) = -W(x)$. Here, $L^2_4(\mathbb{R}) = \{u \in L^2(\mathbb{R}) | x^4 u \in L^2(\mathbb{R})\}$. In this case, it is known that W is bounded relative to H_0; moreover $d = \mathcal{O}(\mathrm{e}^{-1/c\hbar}), D = \mathcal{O}(\hbar), w = \mathcal{O}(1)$ if E_1, E_2 are the two lowest eigenvalues. Hence, the conditions of theorem 1 are fulfilled in the semiclassical regime provided $(e_1|We_2) \neq 0$ and thus there exist $A > 0, B > 0, C > 0$ such that $H_\epsilon(\hbar) := H_0 + \mathrm{i}\epsilon W$ will have at least a pair of complex conjugate eigenvalues for $A\,\mathrm{e}^{-B/\hbar} < \epsilon w \ll C\hbar$. Equivalently, we may consider the double well family $H_0(g) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2(1+gx)^2$ defined on the same domain. Here $d = \mathcal{O}(e^{-1/g^2}), D = \mathcal{O}(1), w = \mathcal{O}(1)$. The same argument holds for the general case $H_0 = -\hbar^2\Delta + V(x)$, where $V : \mathbb{R}^n \to \mathbb{R}$ is smooth, has two equal quadratic minima and diverges positively as $|x| \to \infty$; $W(x) \in L^\infty_{loc}(\mathbb{R}^n), |W(x)| \leqslant AV(x)$ as $|x| \to \infty$ because the estimate for d is the same as above [15].

The second result concerns the opposite situation, a criterion ensuring the reality of the spectrum. In this case the natural assumption is the simplicity of the spectrum of H_0 in addition to its reality. Therefore, for the sake of simplicity we assume H_0 self-adjoint.

Theorem 1.2. *Let the self-adjoint operator H_0 be bounded below (without loss of generality, positive), and let H_1 be continuous. Let H_0 have discrete spectrum, $\sigma(H_0) = \{0 \leqslant \lambda_0 \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_l \leqslant \cdots\}$, with the property*

$$\delta := \inf_{j\geqslant 0}[\lambda_{j+1} - \lambda_j]/2 > 0. \tag{1.5}$$

Then $\sigma(H(\epsilon)) \in \mathbb{R}$ if $\epsilon \in \mathbb{R}$, $|\epsilon| < \frac{\delta}{\|H_1\|}$.

Example. Here, again $\mathcal{H} = L^2(\mathbb{R})$; $H_0 = -\frac{d^2}{dx^2} + V(x)$, $D(H_0) = H^2(\mathbb{R}) \cap D(V)$. $V(x) = kx^{2m}, k > 0, m \geqslant 1$; $W(x) \in L^\infty(\mathbb{R}), W(-x) = -W(x)$. We have: $\sigma(H_0) = \{\lambda_n\}, n = 0, 1, \ldots$;

$$\lambda_n \sim k^{\frac{1}{2m}} n^{\frac{2m}{m+1}}, \qquad n \to \infty$$

Each eigenvalue λ_n is simple. Clearly $\delta \geqslant 1$. Denote now $H_\epsilon := H_0 + \mathrm{i}\epsilon W$ the operator family in $L^2(\mathbb{R})$ defined by $H_\epsilon = H_0 + \epsilon H_1$, $H_1 = \mathrm{i}W$, $D(H_\epsilon) = D(H_0)$. Then H_ϵ has real discrete spectrum for $|\epsilon| < \|W\|_\infty$.

2. Proof of the results

Proof of theorem 1.1. The proof consists in two steps. In the first one we show that the 2×2 matrix generated by restricting the perturbation H^1 to E_{λ_0} is anti-Hermitian in case (i) of theorem 1.1 or Hermitian in case (ii). In the second step we show by the method of the Grushin reduction (see, e.g., [15]) that for ϵ suitably small the control of the above 2×2 matrix is enough to establish the result. A shorter proof of assertion (i) could be obtained by standard first-order degenerate perturbation theory; however, unlike perturbation theory, the Grushin reduction simultaneously yields assertion (ii), so that we limit ourselves to apply perturbation theory to sketch a proof of remark 3 after theorem 1.1.

Let $\{e_1, e_1\}$ be once more a basis in E_{λ_0} such that (1.2) holds, and denote by e_1^*, e_2^* the dual basis in the dual subspace $E_{\lambda_0}^* = JE_{\lambda_0}$. Clearly $Je_j = \tau_j e_j^*, \tau_j = \pm 1$. We denote Π_0 the spectral projection from $\mathcal{H}$ to E_{λ_0}. Explicitly,

$$\Pi_0 u = (u|e_1^*)e_1 + (u|e_2^*)e_2. \tag{2.1}$$

Consider now the rank 2 operator family $\Pi_0 H_\epsilon \Pi_0$ acting on E_{λ_0}. The representing 2×2 matrix is

$$H(\epsilon)_{j,k} = \lambda_0 I + \epsilon H^1_{j,k}, \qquad H^1_{j,k} = (H_1 e_k|e_j^*), \qquad j, k = 1, 2. \tag{2.2}$$

Now $JH_0 = H_0^* J$, $J\Pi_0 = \Pi_0^* J$. We also have $JH_1 = H_1^* J$. Therefore,

$$(JH_1 e_k|e_j) = (H_1^* Je_k|Je_j) = (Je_k|\mathcal{H}_1 e_j) = \tau_j (H_1 e_k|e_j^*) = \tau_j (e_k|e_j^*) = \tau_j H^1_{j,k}$$

and in the same way

$$(JH_1 e_k|e_j) = (H_1^* e_k|e_j) = (Je_k|H_1 e_j) = \tau_k \big(e_k^*|H_j^1\big) = \tau_k \overline{(H_1 e_j|e_k^*)} = \tau_k \overline{H^1_{k,j}}.$$

Summing up,

$$\tau_j H^1_{j,k} = \tau_k \overline{H^1_{k,j}}.$$

Therefore, if $\tau_1\tau_2 = 1$ the matrix $H(\epsilon)_{j,k}$ is Hermitian for $\epsilon \in \mathbb{R}$ and its eigenvalues are real; if instead $\tau_1\tau_2 = -1$ the matrix $H(\epsilon)_{j,k}$ has real diagonal elements and is anti-Hermitian off diagonal for $\epsilon \in \mathbb{R}$; hence its eigenvalues are complex conjugate under condition (1.4). This completes the first step.

We want now to construct an approximate inverse of $H_\epsilon - z$ near λ_0 by solving a Grushin problem. In this context it is equivalent to the Feshbach reduction, and provides a convenient formalism for it. To this end, define the operators R_+, R_-, $\mathcal{P}_0(z)$ in the following way:

$$R_+ : \mathcal{H} \to \mathbb{C}^2, \qquad R_+u(j) = (u|e_j^*), \quad j = 0, 1; \tag{2.3}$$

$$R_- : \mathbb{C}^2 \to \mathcal{H}, \qquad R_-u_- = \sum_{j=1}^{2} u_-(j)e_j, \tag{2.4}$$

$$\mathcal{P}_0(z) = \begin{pmatrix} H_0 - z & R_- \\ R_+ & 0 \end{pmatrix} : \mathcal{D} \times \mathbb{C}^2 \to \mathcal{H} \times \mathbb{C}^2. \tag{2.5}$$

Note that we have identified E_{λ_0} with its representative $\mathbb{C}^2$, and that $R_+R_- = I_d$. The associated Grushin system is

$$\begin{cases} (H_0 - z)u + R_-u_- = f \\ R_+u = f_+ \end{cases} \tag{2.6}$$

where $u \in \mathcal{D}$, $f \in \mathcal{H}$, u_-, $f_+ \in \mathbb{C}^2$. $z \in \mathbb{C}$ belongs to a neighbourhood of λ_0 at a positive distance from $\sigma(H_0)\backslash\{\lambda_0\}$. After determining u_- in such a way that $f - R_-u_- \in (1 - \Pi_0)\mathcal{H}$ the first equation can be solved for $u(z) \in (1 - \Pi_0)\mathcal{H}$ and hence the problem is reduced to the rank 2 equation $R_+u(z) = f_+$. To solve explicitly, remark that, for every z in the complex complement of $\sigma(H_0)\backslash\{\lambda_0\}$, $\mathcal{P}_0(z)$ has the bounded inverse,

$$\mathcal{E}_0(z) = \begin{pmatrix} E^0(z) & E^0_+(z) \\ E^0_-(z) & E^0_{-+}(z) \end{pmatrix}, \tag{2.7}$$

with

$$\begin{aligned} &E^0(z) = (H_0 - z)^{-1}(1 - \Pi_0), \qquad E^0_+(z) = R_-, \\ &E^0_-(z) = R_+, \qquad E^0_{-+}(z) = -zI + \begin{pmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{pmatrix}, \end{aligned} \tag{2.8}$$

where I is the 2×2 identity matrix. The spectral problem within E_{λ_0} is thus reduced to the inversion of $E^0_{-+}(z)$, and obviously its solution is represented by λ_0, e_1, e_2.

Now restrict the attention to the set of complex z with $\mathrm{dist}(z, \{\lambda_0\}) < 1/(2R)$, where

$$R := \|E^0(\lambda_0)\| = \|(1 - \Pi_0)(H_0 - \lambda_0)^{-1}\| \tag{2.9}$$

so that by the geometrical series expansion

$$\|E^0(z)\| \leqslant \frac{R}{1 - |z - \lambda_0|R}. \tag{2.10}$$

Consider the operator from $\mathcal{D} \times \mathbb{C}^2$ to $\mathcal{H} \times \mathbb{C}^2$ defined as

$$\mathcal{P}_\epsilon(z) = \begin{pmatrix} H_\epsilon - z & R_- \\ R_+ & 0 \end{pmatrix}, \tag{2.11}$$

associated with the Grushin system

$$\begin{cases} (H_\epsilon - z)u_1 + R_-u_2 = f_1 \\ R_+u_1 = f_2 \end{cases}. \tag{2.12}$$

Then

$$\mathcal{P}_\epsilon(z)\mathcal{E}_0(z) = 1 + \begin{pmatrix} \epsilon H_1 E^0(z) & \epsilon H_1 E^0_+(z) \\ 0 & 0 \end{pmatrix} =: 1 + \mathcal{K}. \tag{2.13}$$

It is routine to check that $\mathcal{P}_\epsilon(z)$ has the inverse

$$\mathcal{E}_\epsilon(z) = \begin{pmatrix} E^\epsilon(z) & E_+^\epsilon(z) \\ E_-^\epsilon(z) & E_{-+}^\epsilon(z) \end{pmatrix}, \tag{2.14}$$

with

$$E^\epsilon(z) = \sum_{n=0}^{\infty}(-\epsilon)^n E^0(H_1 E^0)^n, \tag{2.15}$$

$$E_+^\epsilon(z) = \sum_{n=0}^{\infty}(-\epsilon)^n (E^0 H_1)^n E_+^0, \tag{2.16}$$

$$E_-^\epsilon(z) = \sum_{n=0}^{\infty}(-\epsilon)^n E_-^0(H_1 E^0)^n, \tag{2.17}$$

$$E_{-+}^\epsilon(z) = E_{-+}^0 + \sum_{n=1}^{\infty}(-\epsilon)^n E_-^0(H_1 E^0)^{n-1} H_1 E_+^0, \tag{2.18}$$

where all the series will be proved to have a positive convergence radius (convergence means here uniform, or, equivalently, in the norm operator sense).

We next derive the appropriate symmetries for the inverse operators. We have

$$J R_- u_- = \sum_{j=1}^{2} u_-(j) J e_j = \sum_{j=1}^{2} (\tau u_-)(j) e_j^*, \qquad \tau := \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}$$

$$R_+^* u_- = \sum_{j=1}^{2} u_-(j) e_j^*$$

where the second equation follows from

$$(u|R_+^* u_-) = \sum_{j=1}^{2} \overline{u_-(j)}(u|e_j^*), \qquad (R_+ u|u_-) = \sum_{j=1}^{2} \overline{u_-(j)}(u|e_j^*).$$

We thus conclude

$$J R_- u_- = R_+^* \tau u_-, \qquad R_-^* J = \tau R_+.$$

Therefore, from $J H_\epsilon = H_\epsilon^* J$ we get

$$\begin{pmatrix} J & 0 \\ 0 & \tau \end{pmatrix} \begin{pmatrix} H_\epsilon - z & R_- \\ R_+ & 0 \end{pmatrix} = \begin{pmatrix} J(H_\epsilon - z) & J R_- \\ \tau R_+ & 0 \end{pmatrix}$$

$$= \begin{pmatrix} (H_\epsilon^* - z)J & R_+^* \tau \\ R_-^* J & 0 \end{pmatrix} = \begin{pmatrix} (H_\epsilon^* - z) & R_+^* \\ R_-^* & 0 \end{pmatrix} \begin{pmatrix} J & 0 \\ 0 & \tau \end{pmatrix}$$

whence

$$\begin{pmatrix} J & 0 \\ 0 & \tau \end{pmatrix} \mathcal{P}_\epsilon(z) = \mathcal{P}_\epsilon(\overline{z})^* \begin{pmatrix} J & 0 \\ 0 & \tau \end{pmatrix}. \tag{2.19}$$

Since $\mathcal{E}(z) = \mathcal{P}(z)^{-1}$, taking right and left inverses we get

$$\mathcal{E}(\overline{z})^* \begin{pmatrix} J & 0 \\ 0 & \tau \end{pmatrix} = \begin{pmatrix} J & 0 \\ 0 & \tau \end{pmatrix} \mathcal{E}(z)$$

that is

$$\begin{pmatrix} E(\overline{z})^* & E_-(\overline{z})^* \\ E_+(\overline{z})^* & E_{-+}(\overline{z})^* \end{pmatrix} \begin{pmatrix} J & 0 \\ 0 & \tau \end{pmatrix} = \begin{pmatrix} J & 0 \\ 0 & \tau \end{pmatrix} \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix}. \tag{2.20}$$

In particular,

$$E_{-+}(\overline{z})^*\tau = \tau E_{-+}(z).$$

We can thus conclude that, for $z \in \mathbb{R}$, if $\tau_1 \cdot \tau_2 = 1$ the 2×2 matrix $E_{-+}(z)$ is Hermitian and anti-Hermitian off diagonal with real diagonal elements if $\tau_1 \cdot \tau_2 = -1$.

It remains to prove the norm convergence of the expansions (2.15), (2.17), (2.18). We have, by the relative boundedness condition $\|H_1\psi\| \leqslant b\|H_0\psi\| + b\|\psi\|$ and (2.10),

$$\begin{aligned}\|H^1E^0\| &= \|H^1(H_0 - z)^{-1}(1 - \Pi_0)\| \\ &\leqslant b\|H_0(H_0 - z)^{-1}(1 - \Pi_0)\| + a\|(H_0 - z)^{-1}(1 - \Pi_0)\| \\ &\leqslant b\|(H_0 - z)(H_0 - z)^{-1}(1 - \Pi_0)\| \\ &\quad + b|z|\|(H_0 - z)^{-1}(1 - \Pi_0)\| + a\|(H_0 - z)^{-1}(1 - \Pi_0)\| \\ &\leqslant b\|1 - \Pi_0\| + \frac{(b|z| + a)R}{1 - |z - \lambda_0|R} < K\end{aligned}$$

for some $K(z) > 0$ because $|z - \lambda_0| < 1/(2R)$. Therefore,

$$\begin{aligned}&\|E^0(H^1E^0)^n\| \leqslant CK^{n+1}, && \|(E^0H^1)^nE^0_+\| \leqslant CK^{n+1}, \\ &\|E^0_-(H^1E^0)^n\| \leqslant CK^{n+1}, && \|E^0_-(H^1E^0)^{n-1}H_1E^0_+\| \leqslant CK^{n+1}.\end{aligned}$$

Hence, the expansions (2.15), (2.17), (2.18) are norm convergent.

To conclude the proof we have to verify that the first-order truncation of the expansion for $E_+(z)$ yields non-real eigenvalues, and that the higher order terms can be neglected. To this end, first remark that without loss of generality we may assume $\lambda_0 = 0$. Then the expansion (2.18) yields (we drop the upper index in H^1_{jk} to simplify the notation)

$$E^\epsilon_{-+}(z) = \begin{pmatrix}\epsilon H_{11} - z & \epsilon H_{12} \\ -\epsilon\overline{H}_{12} & \epsilon H_{22} - z\end{pmatrix} + O(\epsilon^2)$$

uniformly with respect to z, $|z| < 1/2R$. Therefore,

$$\begin{aligned}\det E^\epsilon_{-+}(z) &= z^2 - (H_{11} + H_{22})\epsilon z + (|H_{12}|^2 + H_{11}H_{22})\epsilon^2 + O(\epsilon^3 + \epsilon^2|z|) \\ &= [z - \epsilon(H_{11} + H_{22})/2]^2 + \epsilon^2[|H_{12}|^2 - (H_{11} - H_{22})^2/4] + O(\epsilon^3 + \epsilon^2|z|).\end{aligned}$$

Now $\det E^\epsilon_{-+}(z)$, which is real for $z \in \mathbb{R}$, clearly has no zeros for $z \in \mathbb{C}$, $\epsilon \ll |z| \ll 1$. On the other hand, for $z = O(\epsilon)$, i.e., $z = \epsilon w$, $w = O(1)$,

$$\det E^\epsilon_{-+}(z) = \epsilon^2\{[w - (H_{11} + H_{22})/2]^2 + |H_{12}|^2 - (H_{11} - H_{22})^2/4\} + O(\epsilon^3(1 + O(1)).$$

Therefore, if $4|H_{12}|^2 > (H_{11} - H_{22})^2$ there cannot be real zeros for ϵ suitably small. We can thus conclude that $\det E^\epsilon_{-+}(z)$ is zero for $z = \Lambda_\pm(\epsilon)$,

$$\Lambda_\pm(\epsilon) = \frac{\epsilon}{2}\left[H_{11} + H_{22} \pm \mathrm{i}\sqrt{4|H_{12}|^2 - (H_{11} - H_{22})^2}\,\right] + O(\epsilon^2)$$

and this concludes the proof of the theorem. □

Sketch of the proof of remark 3. Here, E_{λ_0} is replaced by the two-dimensional subspace $\mathcal{E}$ spanned by the eigenvectors ψ_1, ψ_2. Then the first step of the argument can be taken over directly, up to the obvious notational changes, namely, standard first-order perturbation theory entails that up to order ϵ^2 the eigenvalues of H_ϵ around the eigenvalues E_1, E_2 of H_0 are given by the eigenvalues of the 2×2 matrix

$$H^1_{l,k} = \begin{pmatrix}E_1 & \epsilon H_{12} \\ \epsilon H_{21} & E_2\end{pmatrix}, \qquad H_{12} = (\psi_1|H_1\psi_2), \quad H_{21} = (\psi_2|H_1\psi_1)$$

where without loss we have assumed $(\psi_1|H_1\psi_1) = (\psi_2|H_1\psi_2) = 0$. Therefore, $H^1_{l,k}$ will have non-real eigenvalues if $|\epsilon H_{12}| > |E_2 - E_1|/2 = d/2$. This entails that the two eigenvalues of H_ϵ near E_1, E_2 will be likewise non-real as long as the second-order remainder of perturbation theory can be made sufficiently small for ϵ fixed. By standard arguments (see, e.g., [16], chapters II.5 and VII.2) it is enough to control $\|\epsilon H_1 R_0(z)\|$ uniformly in $z \in \Gamma$, where Γ is any circumference encircling E_1, E_2. Choosing as usual $\Gamma := \{z \in \mathbb{C} : |z - E_1| = D/2\}$ where we have assumed without loss E_1 closest to the complement of $\sigma(H_0)$ with respect to $\{E_1, E_2\}$, the following estimate clearly holds:

$$\|\epsilon H_1 R_0(z)\| \leqslant \frac{|\epsilon|\|H_1\|}{\mathrm{dist}(z, \sigma(H_0)} = \frac{|\epsilon|\|H_1\|}{D/2 - d}.$$

Since $|\epsilon H_{12}| < |\epsilon|\|H_1\|$, and the remainder is uniformly small for $\|\epsilon H_1 R_0(z)\| < 1$, we see that the following conditions must hold:

$$\frac{d}{2} < |\epsilon|\|H_1\| < \frac{D}{2} - d.$$

Given $\|H_1\|$, if d/D is small enough there exists $\epsilon^* > 0$ such that this condition holds for all $\epsilon \in [-\epsilon^*, \epsilon^*]$.

Proof of theorem 1.2. Let us first recall that under the present assumptions H_ϵ is a type-A holomorphic family of operators in the sense of Kato (see [16], chapter VII.2) with compact resolvents $\forall \epsilon \in \mathbb{C}$. Hence, $\sigma(H_\epsilon) = \{\lambda_l(\epsilon)\} : l = 0, 1, \ldots$. In particular,

(i) the eigenvalues $\lambda_l(\epsilon)$ are locally holomorphic functions of ϵ with at most algebraic singularities;
(ii) the eigenvalues $\lambda_l(\epsilon)$ are stable, namely given any eigenvalue $\lambda(\epsilon_0)$ of H_{ϵ_0} there is exactly one eigenvalue $\lambda(\epsilon)$ of H_ϵ such that $\lim_{\epsilon\to\epsilon_0} \lambda(\epsilon) = \lambda(\epsilon_0)$;
(iii) the Rayleigh–Schrödinger perturbation expansion for the eigenprojections and the eigenvalues near any eigenvalue λ_l of H_0 has convergence radius $\delta_l/\|H_1\|$ where δ_l is half the isolation distance of λ_l.

Remark that since $\delta_l \geqslant \delta, \forall l$, all the series will be convergent for all $\epsilon \in \Omega_{r_0}$; $\Omega_{r_0} := \{\epsilon \in \mathbb{C} : |\epsilon| \leqslant r_0 < r\}$, where $r := \delta/\|H_1\|$ is a uniform lower bound for all convergence radii.

Assume now without loss of generality, to simplify the notation, $\|H_1\| = 1$. By hypothesis $|\lambda_l - \lambda_{l+1}| \geqslant 2\delta > 0\, \forall l \in \mathbb{N}$. First remark that if $\epsilon \in \mathbb{R}$, $|\epsilon| < r_0$ and $\lambda(\epsilon)$ is an eigenvalue of H_ϵ then $|\mathrm{Im}\,\lambda(\epsilon)| < \delta$, i.e., $\sigma(H_\epsilon) \cap \mathbb{C}_\delta = \emptyset$, $\mathbb{C}_\delta := \{z \in \mathbb{C} | |\mathrm{Im}\, z| \geqslant \delta\}$. Set indeed

$$R_0(z) := [H_0 - z]^{-1}, \qquad z \notin \sigma(H_0).$$

Then $\forall z \in \mathbb{C}$ such that $|\mathrm{Im}\, z| \geqslant \delta$ we have

$$\|\epsilon H_1 R_0(z)\| \leqslant |\epsilon| \cdot \|H_1\| \cdot \|R_0(z)\| \leqslant \frac{|\epsilon|}{\mathrm{dist}[z, \sigma(H_0)]} \leqslant \frac{|\epsilon|}{|\mathrm{Im}\, z|}. \tag{2.21}$$

Hence, the resolvent

$$R_\epsilon(z) := [H_\epsilon - z]^{-1} = R_0(z)[1 + \epsilon H_1 R_0(z)]^{-1}$$

exists and is bounded if $|\mathrm{Im}\, z| \geqslant \delta$ because (2.21) entails the uniform norm convergence of the Neumann expansion for the resolvent:

$$\|R_\epsilon(z)\| = \|[H_\epsilon - z]^{-1}\| = \|R_0(z) \sum_{k=0}^{\infty} [-\epsilon H_1 R_0(z)]^k\|$$
$$\leqslant \|R_0(z)\| \sum_{k=0}^{\infty} |\epsilon^k| \|H_1 R_0(z)]\|^k \leqslant \frac{|\epsilon|}{|\mathrm{Im}\, z| - \epsilon}.$$

Now $\forall l \in \mathbb{N}$ let $Q_l(\delta)$ denote the open square of side 2δ centred at λ_l. Since $|\lambda_l - \lambda_{l+1}| \geqslant 2\delta$, it follows as in (2.21) that $R_\epsilon(z)$ exists and is bounded for $z \in \partial Q_l(\delta)$, the boundary of $Q_l(\delta)$. We can, therefore, according to the standard procedure (see e.g., [16], chapter III.2) define the strong Riemann integrals

$$P_l(\epsilon) = \frac{1}{2\pi \mathrm{i}} \int_{\partial Q_l(\delta)} R_\epsilon(z)\,\mathrm{d}z, \qquad l = 1, 2, \ldots.$$

As is well known, P_l is the spectral projection onto the part of $\sigma(H_\epsilon)$ inside Q_l. Since H_ϵ is a holomorphic family in ϵ, by well-known results (see, e.g., [16], theorem VII.2.1), the same is true for $P_l(\epsilon)$ for all $l \in \mathbb{N}$. In particular, this entails the continuity of $P_l(\epsilon)$ for $|\epsilon| < r_0$. Now $P_l(0)$ is a one dimensional: hence the same is true for $P_l(\epsilon)$. As a consequence, there is one and only one point of $\sigma(H_\epsilon)$ inside any Q_l. Now $\sigma(H_\epsilon)$ is discrete, and thus any such point is an eigenvalue; moreover, any such point is real for ϵ real because $\sigma(H_\epsilon)$ is symmetric with respect to the real axis. Finally, we note that if $z \in \mathbb{R}$, $z \notin \bigcup_{l=1}^{\infty}]\lambda_l - \delta, \lambda_l + \delta[$ the Neumann series (2.21) is convergent and the resolvent $R_\epsilon(z)$ is continuous. This concludes the proof of theorem 1.2. □

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